Darboux Transformation and Symbolic Computation on Multi-Soliton and Periodic Solutions for Multi-Component Nonlinear Schrödinger Equations in an Isotropic Medium

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Z. Naturforsch. **64a,** 300 – 308 (2009); received June 17, 2008 / revised September 15, 2008

The Darboux transformation is applied to a multi-component nonlinear Schrödinger system, which governs the propagation of polarized optical waves in an isotropic medium. Based on the Lax pair associated with this integrable system, the formula for the n-times iterative Darboux transformation is constructed in the form of block matrices. The purely algebraic iterative algorithm is carried out via symbolic computation, and two different kinds of solutions of practical interest, i. e., bright multi-soliton solutions and periodic solutions, are also presented according to the zero and nonzero backgrounds.

Key words: Darboux Transformation; Soliton; Integrable System; Symbolic Computation.

1. Introduction

Integrable nonlinear partial differential equations (PDEs), such as the Korteweg-de Vries (KdV), sine-Gordon (SG) and nonlinear Schrödinger (NLS) equations, have significant mathematical properties and extensive applications in various fields of physics and engineering sciences [1, 2]. An important feature of these integrable PDEs is that they can be expressed as the compatibility conditions for two linear differential equations (Lax pair) with a spectral parameter [1]. The well-known Ablowitz-Kaup-Newell-Segur (AKNS) system gives a systematic procedure to derive a broad class of integrable nonlinear models and solves their initial value problems [3]. Considering more additional physical degrees of freedom, the multi-component nonlinear equations emerging from generalizations of the one-component PDEs are used to describe the nonlinear phenomena under certain physical contexts. For example, the multi-component NLS system generalized from the scalar NLS equation, governs the dynamics of multi-component fields in nonlinear optical fibers [4 – 6]. In the mean-field theory of condensation, the multi-component Gross-Pitaevskii model describes the time evolution of the spinor condensate wavefunction in Bose-Einstein condensates with internal degrees of freedom [7]. Through generalizing the 2×2 linear eigenvalue problems in [3] to 3×3 (even $N\times N$) cases, many multi-component integrable equations of physical significance can be derived; for instance, the modified KdV and NLS equations can be generalized as [8 – 14]

$$u_{j_t}+6\left(\sum_{k,l=1}^{M} C_{kl}\,u_k\,u_l\right)u_{j_x}+u_{j_{xxx}}=0 \quad (j=1,2,\cdots,M), \tag{1}$$

$$\mathrm{i}\,q_{j_z}+q_{j_{tt}}+2\left(\sum_{k=1}^{N}|q_k|^2\right)q_j=0 \quad (j=1,2,\cdots,N). \tag{2}$$

The above two equations both have an infinite number of conservation laws and multi-soliton solutions, and their initial value problems have been solved by the inverse scattering method [10].

We present another multi-component system, i. e., N-coupled nonlinear Schrödinger (N-CNLS) equations

$$\mathrm{i}\,q_{j_z}+q_{j_{tt}}+2\left(|q_j|^2+2\sum_{k=1}^{N}|q_k|^2\right)q_j-2\sum_{k=1}^{N}q_k^2q_j^*=0$$
$$(j,k=1,2,\cdots,N;k\neq j), \tag{3}$$

which can be used to describe the simultaneous propagation of polarized optical waves in an isotropic medium [15–20], where q_j are the varying complex envelopes of optical modes, the variables z and t, respectively, represent the normalized distance along the fiber and the retarded time, and the asterisk denotes the complex conjugate. The last three terms on the left-hand side of System (3) are the self-phase modulation, cross-phase modulation, and coherent energy coupling terms, respectively [18]. This system possesses the Painlevé property, and its Lax pair and conserved quantities have been derived [20]. For $N=2$, the multi soliton-solutions have been constructed and the soliton interaction behaviours have been discussed by virtue of the Darboux transformation [19]. When $N=4$, the one-soliton solution has been presented with the Hirota method in [16].

The Darboux transformation, which was first introduced by Darboux in 1882 [21], has been a very powerful tool and widely used in the soliton theory to construct the exact analytical solutions of integrable nonlinear PDEs including soliton solutions, periodic solutions, and rational solutions [14, 22–25]. In order to apply this method, it is necessary to find the linear eigenvalue problems associated with integrable nonlinear PDEs. Through solving the relevant linear system with a given seed solution, with no need to refer to the special boundary conditions, a series of new analytical solutions can be generated under the Darboux transformation. By performing the iterative algorithm of the Darboux transformation successively, one can obtain the n-times iterated potential transformation formula in terms of the Wronskian determinant [22] or Vandermonde-like determinant [26].

This paper is devoted to applying the Darboux transformation method to System (3) based on the Lax pair derived from the matrix AKNS scheme. To make the iterative algorithm of the Darboux transformation exercisable, we will employ the computerized symbolic computation to deal with a large amount of tedious algebraic calculations. In Section 2, we will briefly review the matrix AKNS system and the Lax pair of System (3) within the framework of block matrices. In Section 3, we will consider how to construct the Darboux transformation of System (3) and get the formula of the n-times iterative Darboux transformation. Subsequently, in Section 4 we will in detail give the iterative procedure of the Darboux transformation and construct the bright multi-soliton solutions and periodic solutions. The last section will be our conclusions.

2. Lax Pairs and Reductions

In this section, we review the previous results of [10] and introduce the AKNS scheme in terms of the block matrices. Considering the linear eigenvalue problems

$$\begin{aligned}&\Psi_t=\mathbf{U}\Psi=(\lambda\,\mathbf{U}_0+\mathbf{U}_1)\,\Psi\ \text{ and}\\ &\Psi_z=\mathbf{V}\Psi=(\lambda^2\,\mathbf{V}_0+\lambda\,\mathbf{V}_1+\mathbf{V}_2)\,\Psi\end{aligned} \tag{4}$$

with the block matrices $\mathbf{U}_0,\mathbf{U}_1,\mathbf{V}_0,\mathbf{V}_1,\mathbf{V}_2$ as

$$\mathbf{U}_0=\mathrm{i}\begin{pmatrix}-\mathbf{I}_1&\mathbf{0}\\ \mathbf{0}&\mathbf{I}_2\end{pmatrix},\quad \mathbf{U}_1=\begin{pmatrix}\mathbf{0}&\mathbf{Q}\\ \mathbf{R}&\mathbf{0}\end{pmatrix}, \tag{5}$$

$$\begin{aligned}&\mathbf{V}_0=2\,\mathrm{i}\begin{pmatrix}-\mathbf{I}_1&\mathbf{0}\\ \mathbf{0}&\mathbf{I}_2\end{pmatrix},\quad \mathbf{V}_1=2\begin{pmatrix}\mathbf{0}&\mathbf{Q}\\ \mathbf{R}&\mathbf{0}\end{pmatrix},\\ &\mathbf{V}_2=\mathrm{i}\begin{pmatrix}-\mathbf{Q}\mathbf{R}&\mathbf{Q}_t\\ -\mathbf{R}_t&\mathbf{R}\mathbf{Q}\end{pmatrix},\end{aligned} \tag{6}$$

where Ψ is a vector function, $\mathbf{I}_1$ and $\mathbf{I}_2$ are, respectively, the $p\times p$ and $m\times m$ unit matrices, $\mathbf{Q}$ and $\mathbf{R}$ are, respectively, the $p\times m$ and $m\times p$ matrices, λ is the spectral parameter independent of z and t, the compatibility condition for (4), i. e., the zero-curvature equation $\mathbf{U}_z-\mathbf{V}_t+[\mathbf{U},\mathbf{V}]=0$, where the brackets denote the commutator of two matrices, yields the coupled matrix equations

$$\mathrm{i}\,\mathbf{Q}_z+\mathbf{Q}_{tt}-2\,\mathbf{Q}\mathbf{R}\mathbf{Q}=\mathbf{0}, \tag{7}$$

$$\mathrm{i}\,\mathbf{R}_z-\mathbf{R}_{tt}+2\,\mathbf{R}\mathbf{Q}\mathbf{R}=\mathbf{0}. \tag{8}$$

Under the reduction

$$\mathbf{R}=-\mathbf{Q}^\dagger, \tag{9}$$

where the sword denotes the Hermitian conjugate, (7) and (8) reduce to the matrix NLS equation

$$\mathrm{i}\,\mathbf{Q}_z+\mathbf{Q}_{tt}+2\,\mathbf{Q}\mathbf{Q}^\dagger\mathbf{Q}=\mathbf{0}, \tag{10}$$

from which the multi-component NLS-type systems are able to be derived according to the different forms of $\mathbf{Q}$.

Case I. $p = 1$, $m = 1$.

For the simple case $\mathbf{Q} = q$, (10) leads to the standard NLS equation

$$\mathrm{i}\,q_z + q_{tt} + 2|q|^2 q = 0. \quad (11)$$

Case II. $p = 1$, $m = N$.

Taking $\mathbf{Q} = (q_1, q_2, \cdots, q_N)$, (10) leads to System (2).

Case III. $p = 2^{N-1}$, $m = 2^{N-1}$.

If $\mathbf{Q}$ is, respectively, chosen as the following forms:

$$\mathbf{Q}_2 = \left(\begin{array}{c|c} q_1 & q_2 \\ \hline -q_2 & q_1 \end{array}\right)_{2\times 2}, \quad (12)$$

$$\mathbf{Q}_3 = \left(\begin{array}{cc|cc} q_1 & q_2 & q_3 & 0 \\ -q_2 & q_1 & 0 & q_3 \\ \hline -q_3 & 0 & q_1 & -q_2 \\ 0 & -q_3 & q_2 & q_1 \end{array}\right)_{4\times 4}, \quad (13)$$

$$\mathbf{Q}_4 = \left(\begin{array}{cccc|cccc} q_1 & q_2 & 0 & 0 & q_3 & 0 & q_4 & 0 \\ -q_2 & q_1 & 0 & 0 & 0 & q_3 & 0 & q_4 \\ 0 & 0 & q_1 & q_2 & -q_4 & 0 & q_3 & 0 \\ 0 & 0 & -q_2 & q_1 & 0 & -q_4 & 0 & q_3 \\ \hline -q_3 & 0 & q_4 & 0 & q_1 & -q_2 & 0 & 0 \\ 0 & -q_3 & 0 & q_4 & q_2 & q_1 & 0 & 0 \\ -q_4 & 0 & -q_3 & 0 & 0 & 0 & q_1 & -q_2 \\ 0 & -q_4 & 0 & -q_3 & 0 & 0 & q_2 & q_1 \end{array}\right)_{8\times 8}, \quad (14)$$

the 2-CNLS, 3-CNLS and 4-CNLS equations of System (3) are obtained by substituting (12)–(14) into (10).

The general expression of $\mathbf{Q}_N$ is the $2^{N-1} \times 2^{N-1}$ block matrix

$$\mathbf{Q}_N = \left(\begin{array}{c|c} \mathbb{Q}_1 & \mathbb{Q}_2 \\ \hline \mathbb{Q}_3 & \mathbb{Q}_4 \end{array}\right), \quad (15)$$

where $\mathbb{Q}_j$ $(j = 1, 2, 3, 4)$ are all $2^{N-2} \times 2^{N-2}$ square-block matrices, $\mathbb{Q}_1$ is a block diagonal matrix, while $\mathbb{Q}_3 = -\mathbb{Q}_2^T$, $\mathbb{Q}_4 = \mathbb{Q}_1^T$ (T denotes the transpose of the matrix). $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are given by

$$\mathbb{Q}_1 = \begin{pmatrix} \mathbb{A}_1 & & & \mathbf{O} \\ & \mathbb{A}_1 & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbb{A}_1 \\ & & & & \mathbb{A}_1 \end{pmatrix}, \quad (16)$$

$$\mathbb{Q}_2 = \left(\begin{array}{c|c} \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{B}_3 & \mathbb{B}_4 \end{array}\right),$$

where $\mathbb{Q}_2$ has the same identities as $\mathbf{Q}_N$, i.e., $\mathbb{B}_j$ are all square-block matrices, $\mathbb{B}_3 = -\mathbb{B}_2^T$, $\mathbb{B}_4 = \mathbb{B}_1^T$, while $\mathbb{B}_1$ and $\mathbb{B}_2$ are expressible in the forms

$$\mathbb{B}_1 = \begin{pmatrix} \mathbb{A}_3 & & \mathbf{O} \\ & \mathbb{A}_3 & \\ \mathbf{O} & \ddots & \\ & & \mathbb{A}_3 \end{pmatrix},$$

$$\mathbb{B}_2 = \begin{pmatrix} \mathbb{A}_4 & \mathbb{O} & \cdots & \mathbb{A}_{N-1} & \mathbb{A}_N \\ \mathbb{O} & \mathbb{A}_4 & \cdots & -\mathbb{A}_N & \mathbb{A}_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathbb{A}_{N-1} & \mathbb{A}_N & \cdots & \mathbb{A}_4 & \mathbb{O} \\ -\mathbb{A}_N & -\mathbb{A}_{N-1} & \cdots & \mathbb{O} & \mathbb{A}_4 \end{pmatrix}, \quad (17)$$

$$\mathbb{A}_1 = \begin{pmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{pmatrix}, \quad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\mathbb{A}_j = \begin{pmatrix} q_j & 0 \\ 0 & q_j \end{pmatrix} \quad (j = 3, 4, \cdots, N). \quad (18)$$

It is easy to verify that System (3) can be derived from (10) with the substitution of (15)–(18).

In terms of Lie algebraic structures, the various reductions of the Lax pair associated with the matrix NLS equation have been investigated in the literatures [12, 27]. Using the generators of the Clifford algebra, a class of reductions of the matrix NLS equation were considered in [27]. Many other generalizations investigated in [12] are associated with the Hermitian symmetric space.

3. The Darboux Transformation of System (3)

In this section, let us now turn our attention to how to construct the Darboux transformation of System (3). We introduce the following transformation:

$$\boldsymbol{\Psi}[1] = \mathbf{D}\boldsymbol{\Psi} = (\lambda\,\mathbf{I} - \mathbf{S})\boldsymbol{\Psi}, \quad (19)$$

where $\boldsymbol{\Psi}$ is a 2^N-dimensional vector function, $\mathbf{I}$ is a $2^N \times 2^N$ unit matrix, and $\mathbf{D}$ is called the Darboux matrix. It requires that $\boldsymbol{\Psi}[1]$ should still satisfy the same linear eigenvalue problems (4):

$$\begin{aligned} \boldsymbol{\Psi}[1]_t &= (\lambda\,\mathbf{U}_0[1] + \mathbf{U}_1[1])\boldsymbol{\Psi}[1], \\ \boldsymbol{\Psi}[1]_z &= (\lambda^2\,\mathbf{V}_0[1] + \lambda\,\mathbf{V}_1[1] + \mathbf{V}_2[1])\boldsymbol{\Psi}[1], \end{aligned} \quad (20)$$

with $\mathbf{U}_0[1], \mathbf{U}_1[1], \mathbf{V}_0[1], \mathbf{V}_1[1], \mathbf{V}_2[1]$ having the same forms as (5) and (6) with $\mathbf{Q}$ replaced by $\mathbf{Q}[1]$ (q_j replaced by $q_j[1]$). The compatibility condition $\boldsymbol{\Psi}[1]_{tz} =$

$\Psi[1]_{zt}$ yields the following set of equations:

$$\mathbf{U}_0[1] = \mathbf{U}_0, \quad \mathbf{V}_0[1] = \mathbf{V}_0, \tag{21}$$

$$\mathbf{U}_1[1] - \mathbf{U}_1 + \mathbf{S}\mathbf{U}_0 - \mathbf{U}_0\mathbf{S} = \mathbf{0}, \tag{22}$$

$$\mathbf{U}_1[1]\,\mathbf{S} - \mathbf{S}\mathbf{U}_1 - \mathbf{S}_t = \mathbf{0}, \tag{23}$$

$$\mathbf{V}_1[1] - \mathbf{V}_1 + \mathbf{S}\mathbf{V}_0 - \mathbf{V}_0\mathbf{S} = \mathbf{0}, \tag{24}$$

$$\mathbf{V}_2[1] - \mathbf{V}_2 + \mathbf{S}\mathbf{V}_1 - \mathbf{V}_1[1]\mathbf{S} = \mathbf{0}, \tag{25}$$

$$\mathbf{V}_2[1]\,\mathbf{S} - \mathbf{S}\mathbf{V}_2 - \mathbf{S}_z = \mathbf{0}. \tag{26}$$

From (21) it is shown that the highest components in λ are invariant under the Darboux transformation.

Note that if $\mathbf{S}$ satisfies (22)–(26), the linear eigenvalue problems (4) are invariant after the action of the Darboux transformation (19). The remaining problem is to construct the matrix $\mathbf{S}$ by virtue of the solutions of the linear eigenvalue problems (4). However, there are reduction (9) and constraints among elements in $\mathbf{Q}$ and the Darboux transformation should keep the reduction and those constraints. Therefore, it is not easy to treat the reduced problem. In the 2×2 AKNS system with *U(N)* reduction, the Darboux transformations for one-component cases like KdV, SG and NLS equations have been derived in [22, 23]. Recently, this problem for some multi-component equations has been discussed with the aid of the projection operator [14, 24]. Motivated by the previous work, the construction of $\mathbf{S}$ is in the form

$$\begin{aligned}
&\mathbf{S} = \mathbf{H}\Lambda\mathbf{H}^{-1} = \left(\begin{array}{c|c} \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_3 & \mathbf{S}_4 \end{array}\right), \\
&\Lambda = \left(\begin{array}{c|c} \lambda_1\mathbf{I} & \mathbf{O} \\ \hline \mathbf{O} & \lambda_1^*\mathbf{I} \end{array}\right),
\end{aligned} \tag{27}$$

where $\mathbf{H} = (\Psi,\Phi), \Psi = (\Psi_1,\Psi_2,\cdots,\Psi_{2^{N-1}}), \Phi = (\Phi_1,\Phi_2,\cdots,\Phi_{2^{N-1}})$, Ψ_j and Φ_j $(j=1,2,\cdots,2^{N-1})$ are 2^N-dimensional column vectors, $\mathbf{S}_j$ $(j=1,2,3,4)$ are all $2^{N-1}\times 2^{N-1}$ matrices, and $\mathbf{I}$ is a $2^{N-1}\times 2^{N-1}$ unit matrix.

Back to (22) and (24), $\mathbf{S}$ has to satisfy the reduction condition

$$\mathbf{S}_3 = -\mathbf{S}_2^\dagger. \tag{28}$$

In what follows, we first present the parts of Ψ for 2-CNLS, 3-CNLS and 4-CNLS equations for System (3):

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & -\psi_1 \\ \psi_3 & \psi_4 \\ \psi_4 & -\psi_3 \end{pmatrix}, \tag{29}$$

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_2 & -\psi_1 & -\psi_4 & \psi_3 \\ \psi_3 & \psi_4 & -\psi_1 & -\psi_2 \\ \psi_4 & -\psi_3 & \psi_2 & -\psi_1 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \\ \psi_6 & -\psi_5 & -\psi_8 & \psi_7 \\ \psi_7 & \psi_8 & -\psi_5 & -\psi_6 \\ \psi_8 & -\psi_7 & \psi_6 & -\psi_5 \end{pmatrix}, \tag{30}$$

$$\Psi = \begin{pmatrix}
\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 & \psi_8 \\
\psi_2 & -\psi_1 & \psi_4 & -\psi_3 & -\psi_6 & \psi_5 & -\psi_8 & \psi_7 \\
\psi_3 & \psi_4 & -\psi_1 & -\psi_2 & -\psi_7 & -\psi_8 & \psi_5 & \psi_6 \\
\psi_4 & -\psi_3 & -\psi_2 & \psi_1 & \psi_8 & -\psi_7 & -\psi_6 & \psi_5 \\
\psi_5 & \psi_6 & \psi_7 & \psi_8 & -\psi_1 & -\psi_2 & -\psi_3 & -\psi_4 \\
\psi_6 & -\psi_5 & \psi_8 & -\psi_7 & \psi_2 & -\psi_1 & \psi_4 & -\psi_3 \\
\psi_7 & \psi_8 & -\psi_5 & -\psi_6 & \psi_3 & \psi_4 & -\psi_1 & -\psi_2 \\
\psi_8 & -\psi_7 & -\psi_6 & \psi_5 & -\psi_4 & \psi_3 & \psi_2 & -\psi_1 \\
\psi_9 & \psi_{10} & \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} \\
\psi_{10} & -\psi_9 & \psi_{12} & -\psi_{11} & -\psi_{14} & \psi_{13} & -\psi_{16} & \psi_{15} \\
\psi_{11} & \psi_{12} & -\psi_9 & -\psi_{10} & -\psi_{15} & -\psi_{16} & \psi_{13} & \psi_{14} \\
\psi_{12} & -\psi_{11} & -\psi_{10} & \psi_9 & \psi_{16} & -\psi_{15} & -\psi_{14} & \psi_{13} \\
\psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} & -\psi_9 & -\psi_{10} & -\psi_{11} & -\psi_{12} \\
\psi_{14} & -\psi_{13} & \psi_{16} & -\psi_{15} & \psi_{10} & -\psi_9 & \psi_{12} & -\psi_{11} \\
\psi_{15} & \psi_{16} & -\psi_{13} & -\psi_{14} & \psi_{11} & \psi_{12} & -\psi_9 & -\psi_{10} \\
\psi_{16} & -\psi_{15} & -\psi_{14} & \psi_{13} & -\psi_{12} & \psi_{11} & \psi_{10} & -\psi_9
\end{pmatrix}. \tag{31}$$

Similarly, the framework of Ψ for an N-CNLS system can be obtained inductively by

$$\Psi = \begin{pmatrix}
\psi_1 & \psi_2 & \cdots & \psi_{2^{N-1}-1} & \psi_{2^{N-1}} \\
\psi_2 & -\psi_1 & \cdots & -\psi_{2^{N-1}} & \psi_{2^{N-1}-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\psi_{2^{N-1}-1} & \psi_{2^{N-1}} & \cdots & -\psi_1 & -\psi_2 \\
\psi_{2^{N-1}} & -\psi_{2^{N-1}-1} & \cdots & \psi_2 & -\psi_1 \\
\psi_{2^{N-1}+1} & \psi_{2^{N-1}+2} & \cdots & \psi_{2^N-1} & \psi_{2^N} \\
\psi_{2^{N-1}+2} & -\psi_{2^{N-1}+1} & \cdots & -\psi_{2^N} & \psi_{2^N-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\psi_{2^N-1} & \psi_{2^N} & \cdots & -\psi_{2^{N-1}+1} & -\psi_{2^{N-1}+2} \\
\psi_{2^N} & -\psi_{2^N-1} & \cdots & \psi_{2^{N-1}+2} & -\psi_{2^{N-1}+1}
\end{pmatrix}. \tag{32}$$

The column vectors Ψ_i are 2^{N-1} linearly independent solutions of the linear eigenvalue problem (4) with the spectral parameter λ_i, and the column vectors Φ_j are 2^{N-1} linearly independent solutions of the linear eigenvalue problem (4) with λ_j replaced by λ_j^*. At the same time, the vectors Φ_j $(j=1,2,\cdots,2^{N-1})$ have to satisfy the orthogonality condition $\langle \Psi_i \mid \Phi_j \rangle \equiv 0$ [14, 24].

It is easy to check that $\mathbf{S}$ defined by (27) satisfies (22)–(26). To this stage, we can draw the conclusion that the linear eigenvalue problems (4) associated with System (3) are kept invariant under transformation (19). From (22) and (24), the relationship between $\mathbf{Q}$ and $\mathbf{Q}[1]$ is expressed as

$$\mathbf{Q}[1] = \mathbf{Q} - 2\,\mathrm{i}\,\mathbf{S}_2, \tag{33}$$

from which we see that the new solution $(q_1[1], q_2[1], \cdots, q_N[1])$ is gotten starting from the seed solution $(q_1, q_2, \cdots, q_N)$.

By applying the Darboux transformation successively and taking 2^{N-1} vector solutions of the linear eigenvalue problems (4) with different spectral parameters $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, the n-times iteration of the Darboux transformation is in the form

$$\begin{aligned}\mathbf{D}_n &= (\lambda - \mathbf{S}[n])(\lambda - \mathbf{S}[n-1])\cdots(\lambda - \mathbf{S}[1])\\ &= \lambda^n \mathbf{I} + \sum_{j=0}^{n-1} \mathbf{T}_j \lambda^j,\end{aligned} \tag{34}$$

$$\begin{aligned}&\mathbf{Q}[n] = \mathbf{Q} + 2\,\mathrm{i}\,(\mathbf{T}_{n-1})_2,\\ &\mathbf{T}_{n-1} = -(\mathbf{S}[1] + \mathbf{S}[2] + \cdots + \mathbf{S}[n]),\end{aligned} \tag{35}$$

where $\mathbf{S}[k] = \mathbf{H}[k]\Lambda_k\mathbf{H}^{-1}[k]$ $(k = 1, 2, \cdots, n)$, $\mathbf{T}_{n-1}$ is a block matrix like $\mathbf{S}$, and its elements are defined as [19, 23, 28]

$$\begin{aligned}&(\mathbf{T}_{n-1})_{pq} = \frac{1}{\det(\mathrm{W_n})}\begin{vmatrix} & 0\\ W_n & \vdots\\ & 0\\ & \mu_q\\ h_p^{(n)} & 0\end{vmatrix},\\ &W_n = \begin{pmatrix} H_1 & H_2 & \cdots & H_n\\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_n\Lambda_n\\ \vdots & \vdots & & \vdots\\ H_1\Lambda_1^{n-1} & H_2\Lambda_2^{n-1} & \cdots & H_n\Lambda_n^{n-1}\end{pmatrix},\end{aligned} \tag{36}$$

where $\Lambda_k = \mathrm{diag}(\lambda_k\mathbf{I}, \lambda_k^*\mathbf{I})$, $h_p^{(n)}$ $\left(1 \leqslant p \leqslant 2^N\right)$ is the p-th row of $(H_1\Lambda_1^n, H_2\Lambda_2^n, \cdots, H_n\Lambda_n^n)$, and μ_q $\left(1 \leqslant q \leqslant 2^N\right)$ is a 2^N-dimensional column vector with all entries as zero except for the identity element in the q-th row.

4. Symbolic Computation on the Multi-Soliton and Periodic Solutions

Although the obvious advantage of the Darboux transformation is that its iterative algorithm is purely algebraic and does not include the integral and differential calculations, the increase of iterative times will bring about a large amount of tedious algebraic calculations which are unmanageable manually. Symbolic computation, as a new branch of artificial intelligence [29], is able to drastically increase the ability of a computer to exactly and algorithmically deal with these problems. Combining the iterative algorithm with symbolic computation, the application of the Darboux transformation to generate new solutions of System (3) includes the following steps:

(1) Solve the linear system (4) with the given seed solution $(q_1, q_2, \cdots, q_N)$ and different spectral parameters λ_m $(m = 1, 2, \cdots, n)$, then obtain linear independent column vector solutions $(\psi_1[\lambda_m], \psi_2[\lambda_m], \cdots, \psi_{2^N}[\lambda_m])^T$.

(2) Substitute the above vector solutions into $H_m[\lambda_m]$, then work out the matrix $\mathbf{T}_{n-1}$ by performing symbolic computation on (36).

(3) Symbolically compute the new solutions by virtue of (35).

In what follows, we will apply the iterative algorithm to construct the bright multi-soliton solutions and periodic solutions.

4.1. Multi-Soliton Solutions

Taking $q_j = 0$ $(j = 1, 2, \cdots, N)$ as the initial solution of System (3) and solving the linear system (4), we get

$$\psi_l[\lambda_m] = c_{lm}\,\mathrm{e}^{-2\,\mathrm{i}\,\lambda_m^2 z - \mathrm{i}\,\lambda_m t} \quad (1 \leqslant l \leqslant 2^{N-1}), \tag{37}$$

$$\psi_h[\lambda_m] = c_{hm}\,\mathrm{e}^{2\,\mathrm{i}\,\lambda_m^2 z + \mathrm{i}\,\lambda_m t} \quad (2^{N-1} + 1 \leqslant h \leqslant 2^N), \tag{38}$$

where c_{lm} and c_{hm} $(m = 1, 2, \cdots, n)$ are all arbitrary complex constants. According to (35), the explicit representation of the n-soliton solution can be obtained.

4.2. Periodic Solutions

Starting from the nontrivial solutions

$$\begin{aligned}&q_j = A_j\,\mathrm{e}^{-\mathrm{i}\,\theta}, \quad \theta = \zeta\,t + \Big(\zeta^2 - 2\sum_{k=1}^{N} A_k^2\Big)\,z\\ &(j = 1, 2, \cdots, N),\end{aligned} \tag{39}$$

where A_j are arbitrary nonzero real constants, and ζ is an arbitrary real constant. Based on the investigation on the one-component NLS equation [22], we consider the following two cases through solving the linear system (4).

Case I. When the eigenvalues $\kappa_1 = \kappa_2 = 0$, the characteristic equation is

$$(\zeta - 2\lambda)^2 + 4\sum_{k=1}^{N} A_k^2 = 0, \tag{40}$$

and

$$\psi_l[\lambda_m] = [\alpha_m t + (\zeta + 2\lambda_m)\alpha_m z + \delta_m]\,\mathrm{e}^{-\mathrm{i}\theta/2} \quad (1 \leqslant l \leqslant 2^{N-1}), \tag{41}$$

$$\psi_h[\lambda_m] = -\frac{\mathrm{i}\, b_{hm}[(\zeta - 2\lambda)\delta_m + \chi_{1m}\alpha_m]}{2\sum_{k=1}^{N} A_k^2}\,\mathrm{e}^{\mathrm{i}\theta/2} \quad (2^{N-1}+1 \leqslant h \leqslant 2^N), \tag{42}$$

with

$$\chi_{1m} = \zeta^2 z + \zeta t - 4\lambda_m^2 - 2\lambda_m t + 2\mathrm{i} \quad (m = 1, 2, \cdots, n),$$

where b_{hm} are the linear combination of A_j $(j = 1, 2, \cdots, N)$ to be determined by solving the linear system (4), and α_m, δ_m are arbitrary complex constants.

Case II. When the eigenvalue $\kappa \neq 0$, the characteristic equation is

$$4\kappa^2 + (\zeta - 2\lambda)^2 + 4\sum_{k=1}^{N} A_k^2 = 0, \tag{43}$$

and

$$\psi_l[\lambda_m] = (\beta_{1m}\mathrm{e}^{\vartheta} + \beta_{2m}\mathrm{e}^{-\vartheta})\,\mathrm{e}^{-\mathrm{i}\theta/2} \quad (1 \leqslant l \leqslant 2^{N-1}), \tag{44}$$

$$\psi_h[\lambda_m] = \frac{\mathrm{i}\, b_{hm}\left(\beta_{1m}\chi_{2m}\mathrm{e}^{\vartheta} - \beta_{2m}\chi_{3m}\mathrm{e}^{-\vartheta}\right)}{2\sum_{k=1}^{N} A_k^2}\,\mathrm{e}^{\mathrm{i}\theta/2} \quad (2^{N-1}+1 \leqslant h \leqslant 2^N), \tag{45}$$

with

$$\begin{aligned} \vartheta &= \kappa[(\zeta + 2\lambda_m)z + t], \\ \chi_{2m} &= -\zeta + 2\lambda_m - 2\mathrm{i}\kappa, \\ \chi_{3m} &= \zeta - 2\lambda_m - 2\mathrm{i}\kappa, \end{aligned} \tag{46}$$

where β_{1m} and β_{2m} are arbitrary complex constants.

The study of the stability of plane waves for the NLS equation is of particular importance in applications of nonlinear optics. The exact plane wave solution for focusing the NLS equation (11) is well-known to be linear unstable [30, 32]. The detailed procedure of linear stability analysis of the plane wave solution has been presented in [30]. Under a similar stability criterion, it is possible to extend the linear stability analysis to multi-component NLS equations, and it is found that the periodic solutions remain unstable.

5. Example: 2-CNLS Equations

In this section, we take System (3) with $N = 2$ as an example to construct the bright multi-soliton solutions and periodic solutions according to the iterative algorithm of the Darboux transformation.

First, we choose Φ_1 and Φ_2 which are orthogonal to Ψ_1 and Ψ_2 as

$$\Phi_1 = \begin{pmatrix} \psi_3^* \\ -\psi_4^* \\ -\psi_1^* \\ \psi_2^* \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \psi_4^* \\ \psi_3^* \\ -\psi_2^* \\ -\psi_1^* \end{pmatrix}. \tag{47}$$

Then, the matrix $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3^* & \psi_4^* \\ \psi_2 & -\psi_1 & -\psi_4^* & \psi_3^* \\ \psi_3 & \psi_4 & -\psi_1^* & -\psi_2^* \\ \psi_4 & -\psi_3 & \psi_2^* & -\psi_1^* \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^* & 0 \\ 0 & 0 & 0 & \lambda_1^* \end{pmatrix} \cdot \begin{pmatrix} \psi_1 & \psi_2 & \psi_3^* & \psi_4^* \\ \psi_2 & -\psi_1 & -\psi_4^* & \psi_3^* \\ \psi_3 & \psi_4 & -\psi_1^* & -\psi_2^* \\ \psi_4 & -\psi_3 & \psi_2^* & -\psi_1^* \end{pmatrix}^{-1}. \tag{48}$$

In order to obtain the multi-soliton solutions, we solve the linear system (4) with the trivial solution $q_1 = 0, q_2 = 0$ and obtain

$$\Psi_1[\lambda_1] = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} c_{11}\,\mathrm{e}^{-2\mathrm{i}\lambda_1^2 z - \mathrm{i}\lambda_1 t} \\ c_{21}\,\mathrm{e}^{-2\mathrm{i}\lambda_1^2 z - \mathrm{i}\lambda_1 t} \\ c_{31}\,\mathrm{e}^{2\mathrm{i}\lambda_1^2 z + \mathrm{i}\lambda_1 t} \\ c_{41}\,\mathrm{e}^{2\mathrm{i}\lambda_1^2 z + \mathrm{i}\lambda_1 t} \end{pmatrix}, \tag{49}$$

where c_{11}, c_{21}, c_{31}, and c_{41} are all arbitrary complex constants.

By substitution of (49) into (35), the one-soliton solution of system (3) can be generated as

$$q_1 = \frac{4\nu_1 \mathrm{i}(\lambda_1^* - \lambda_1)\mathrm{e}^{3\xi_1 + \xi_1^* + \eta_1}\cosh[\xi_1 + \xi_1^* + \eta_1]}{2\nu_2\mathrm{e}^{3(\xi_1+\xi_1^*)+\eta_2}\cosh[\xi_1 + \xi_1^* + \eta_2] + \gamma_1}, \tag{50}$$

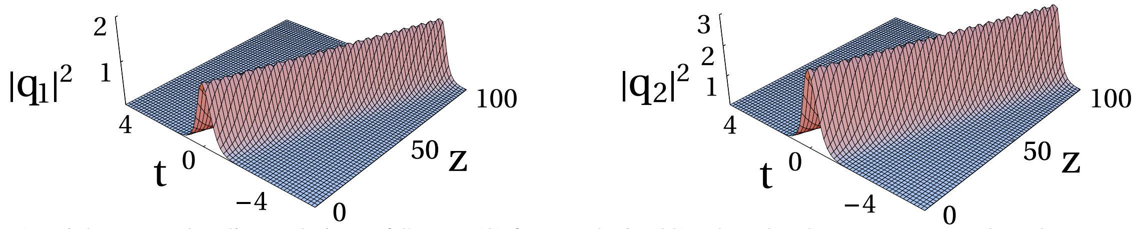

Fig. 1. Bright one-peak soliton solutions of System (3) for $N=2$ via (33). The related parameters are selected as: $c_{11}=1$, $c_{21}=2$, $c_{31}=-2$, $c_{41}=4$, and $\lambda_1=0.01-\mathrm{i}$.

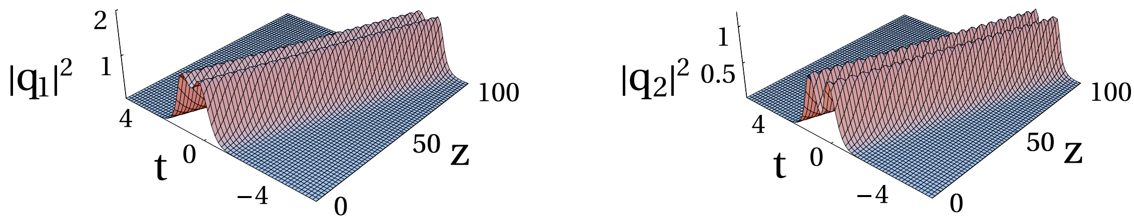

Fig. 2. Bright two-peak soliton solutions of System (3) for $N=2$ via (33). The related parameters are selected as: $c_{11}=5-3\mathrm{i}$, $c_{21}=2+3\mathrm{i}$, $c_{31}=2$, $c_{41}=1-\mathrm{i}$, and $\lambda_1=0.01-\mathrm{i}$.

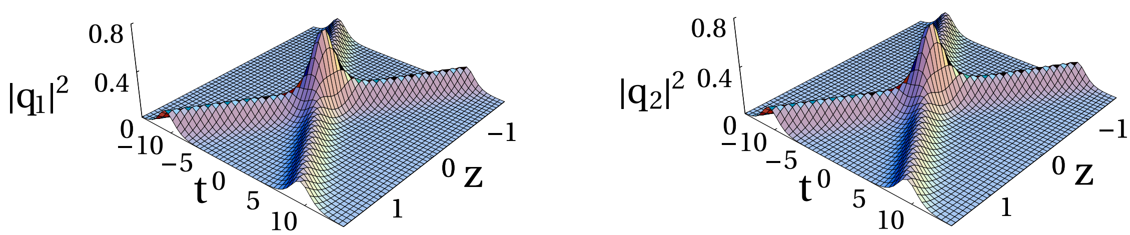

Fig. 3. Interaction between two bright solitons of System (3) for $N=2$ via (35). The relevant parameters are chosen as: $c_{11}=0$, $c_{21}=1$, $c_{31}=1$, $c_{41}=1$, $\lambda_1=-5-0.3\mathrm{i}$, $c_{12}=0$, $c_{22}=1$, $c_{32}=1$, $c_{42}=1$, and $\lambda_2=5-0.3\mathrm{i}$.

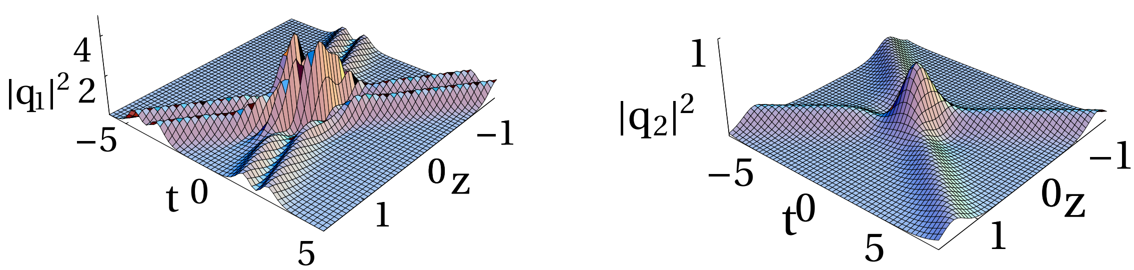

Fig. 4. Interaction between two bright solitons of System (3) for $N=2$ via (35). The relevant parameters are chosen as: $c_{11}=0$, $c_{21}=-5-4\mathrm{i}$, $c_{31}=-6-\mathrm{i}$, $c_{41}=-6\mathrm{i}$, $\lambda_1=2-\mathrm{i}$, $c_{12}=0$, $c_{22}=-4-4\mathrm{i}$, $c_{32}=-6-\mathrm{i}$, $c_{42}=-5\mathrm{i}$ and $\lambda_2=-1+\mathrm{i}$.

$$q_2=\frac{4\nu_3\mathrm{i}(\lambda_1-\lambda_1^*)\mathrm{e}^{3\xi_1+\xi_1^*+\eta_3}\cosh[\xi_1+\xi_1^*+\eta_3]}{2\nu_2\mathrm{e}^{3(\xi_1+\xi_1^*)+\eta_2}\cosh[\xi_1+\xi_1^*+\eta_2]+\gamma_1}, \tag{51}$$

where $\nu_1=(c_{31}^{2*}+c_{41}^{2*})(c_{11}c_{31}+c_{21}c_{41})$, $\nu_2=(c_{11}^2+c_{21}^2)(c_{11}^{2*}+c_{21}^{2*})$, $\nu_3=(c_{31}^{2*}+c_{41}^{2*})(c_{21}c_{31}-c_{11}c_{41})$, $\xi_1=-2\mathrm{i}\lambda_1^2 z-\mathrm{i}\lambda_1 t$, $\mathrm{e}^{2\eta_1}=(c_{11}^2+c_{21}^2)(c_{11}^*c_{31}^*+c_{21}^*c_{41}^*)/\nu_1$, $\mathrm{e}^{2\eta_2}=\nu_2/[(c_{11}c_{11}^*+c_{21}c_{21}^*)(c_{31}c_{31}^*+c_{41}c_{41}^*)+(c_{21}c_{11}^*-c_{11}c_{21}^*)(c_{41}c_{31}^*-c_{31}c_{41}^*)]$, $\mathrm{e}^{2\eta_3}=(c_{11}^2+c_{21}^2)(c_{21}c_{31}-c_{11}c_{41})/\nu_3$, $\gamma_1=(c_{31}^2+c_{41}^2)(c_{31}^{2*}+c_{41}^{2*})$.

In the following, the evolution of the bright one-peak soliton in the respective component is shown in Fig. 1, whilst the bright soliton with two-peak profile is pictured in Figure 2. The significant interest and dynamical behaviours in optical communications for the

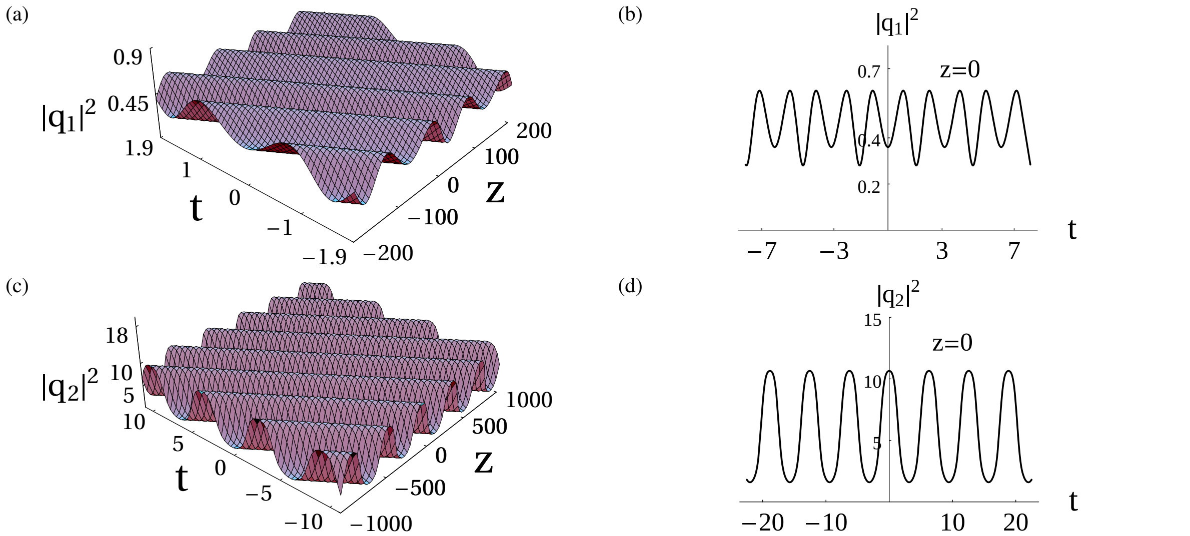

Fig. 5. Periodic wave solutions of System (3) for $N = 2$ via (33). (b) and (d) are the plots corresponding to (a) and (c) at $z = 0$. The relevant parameters are chosen as: $A_1 = 1$, $A_2 = -1$, $\kappa = 0$, $\zeta = 1$, $\delta_1 = 1$, $\alpha_1 = 0$, $\lambda_1 = \frac{1}{2}\left(1+2\mathrm{i}\sqrt{2}\right)$, $b_{31} = A_1 - A_2$, and $b_{41} = A_1 + A_2$.

bright soliton with a two-peak-shaped profile are reported in [19, 33].

With the use of the two sets of basic solutions $(\psi_l[\lambda_1], \psi_h[\lambda_1])$ and $(\psi_l[\lambda_2], \psi_h[\lambda_2])$ of the linear system (4), we can obtain the two-soliton solution of System (3). Figure 3 shows the head-on collision between two bright one-peak solitons in the respective component. They undergo the elastic collision preserving the respective original shapes and velocities, except for the visible phase shifts. As seen in Fig. 4, in the first component two bright two-peak solitons collide elastically with each other, while the second component displays the elastic collisions between two bright one-peak solitons.

Figure 5 shows a family of periodic solutions of System (3) through one-time iteration of the Darboux transformation. The multi-soliton solutions on the periodic background describing periodic modulation of multi-exultons can be generated by substitution of (44) and (45) into (35).

6. Conclusions

There has been much interest in integrable multi-component nonlinear PDEs which can be used to describe various nonlinear phenomena or mechanisms in many fields of physical and engineering sciences. In the present paper, we have shown how the Darboux transformation is applied to a multi-component nonlinear Schrödinger system governing the simultaneous propagation of polarized optical waves in an isotropic medium. Based on the Lax pair derived through the matrix AKNS system in terms of the block matrices, we have constructed the n-times iterative formula by applying the Darboux transformation successively. With the help of symbolic computation, the iterative algorithm of the Darboux transformation can be easily carried out via the iterative determinant representation. We have also constructed the multi-soliton and periodic solutions of this multi-component system. In fact, many more complicated explicit solutions of System (3) can be uncovered with the use of the Darboux transformation. In addition, this algebraic iterative algorithm to construct different kinds of solutions of practical interest is able to be applied to other integrable multi-component equations.

Acknowledgements

We express our sincere thanks to the referees, Prof. Y. T. Gao, and each member of our discussion group for their valuable comments. We also thank Prof. M. Wadati for providing the related literature. This work has been supported by the National Natural Science Foundation of China under Grant No. 60772023, by the Open Fund of the State Key Laboratory of Software Development Environment, Beijing University of Aeronautics and Astronautics under Grant No. SKLSDE-07-001, by the National Basic Research Program of China (973 Program) under Grant No. 2005CB321901, and by the Specialized Research Fund for the Doctoral Program of Higher Education, Chinese Ministry of Education (Nos. 20060006024 and 200800130006).

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